AU-6242

M.A./M.Sc. (Final) Examination, 2014

Mathematics

Paper: First (compulsory) 3rd Semester

Integration Theory

Time: 3 Hrs

MODEL ANSWERS

I a) Let ν and μ be measure functions defined on a space $(X, \mathcal{A})$. The measure ν is said to be absolutely continuous w.r.t. μ iff $\mu(A)=0 \ \forall A \in \mathcal{A} \Rightarrow \nu(A)=0$ and is denoted by $\nu \ll \mu$.

b) since $f = f^+ - f^-$, so $\phi(E) = \int_E f\,d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$

we have either $\int_E f^+ d\mu < \infty$ or $\int_E f^- d\mu < \infty$ so that ϕ takes at most one of the values $\infty, -\infty$ and $\phi(\phi) = 0$ is trivial.

Let $\langle E_i \rangle$ be a sequence of disjoint sets of β and for $E \in \beta$ write $\phi^+(E) = \int_E f^+ d\mu$, $\phi^-(E) = \int_E f^- d\mu$, so that ϕ^+ and ϕ^- are measures. Then

$$\phi\left(\bigcup_{i=1}^{\infty} E_i\right) = \phi^+(\cup E_i) - \phi^-(\cup E_i)$$
$$= \Sigma\, \phi^+(E_i) - \Sigma\, \phi^-(E_i) = \Sigma\, \phi(E_i)$$

$\therefore$ ϕ is a signed measure

c) <u>Lebesgue Decomposition</u>: Let (X, β, μ) be a σ-finite measure space and ν a σ-finite measure defined on β. Then we can find a measure ν_0 which is singular w.r.t μ and a measure ν_1 which is absolutely continuous w.r.t μ such that $\nu = \nu_0 + \nu_1$. The measures ν_0 and ν_1 are unique.

d) <u>Fubini's theorem</u>: Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two measure spaces and ν be complete. Let f be integrable over $X \times Y$ w.r.t the product measure $\mu \times \nu$. Then for almost all $x \in X$, the x-section of f, $f(x, \cdot)$, is integrable over Y w.r.t ν and

$$\int_{X \times Y} f\, d(\mu \times \nu) = \int_X \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x).$$

e) The value of $D^+f(0)=1$ and the value of $D^-f(0)=1$

(calculation needed)

f) Let f be continuous at x_0. Then for every $\varepsilon>0$, there is a $\delta>0$ such that

$$|f(t)-f(x_0)|<\varepsilon, \quad \text{for } |t-x_0|<\delta$$

For $|h|<\delta$, we have

$$\frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt < \varepsilon.$$

g) If f is a fn of bounded variation on $[a,b]$, then the function $V_f:[a,b]\to\mathbb{R}$ defined by

$$V_f(x)=T_a^x(f) \quad [\text{variation fn}]$$

$$V_f(x)\le T_a^b(f) \quad \text{for } a\le x\le b.$$

If x_1, x_2 are two points of $[a,b]$ s.t $x_2>x_1$, then

$$0\le |f(x_2)-f(x_1)| \le T_{x_1}^{x_2}(f)$$
$$= T_a^{x_2}(f)-T_a^{x_1}(f)$$
$$= V_f(x_2)-V_f(x_1) \Rightarrow V_f(x_2)\ge V_f(x_1).$$

h) Let $f(x)$ and $g(x)$ be absolutely continuous fn over the closed interval $[a,b]$ so that given $\varepsilon>0$, $\exists\ \delta>0$ s.t

$$\sum_{k=1}^{n} |f(b_k)-f(a_k)|<\varepsilon \text{ and } \sum_{k=1}^{n} |g(b_k)-g(a_k)|<\varepsilon$$

whenever $\sum_{k=1}^{n}(b_k-a_k)<\delta$ $\forall\ a_1,b_1,a_2,b_2,\dots,a_n,b_n$ s.t

$$a_1<b_1\le a_2<b_2\le\dots\le a_n<b_n.$$

Let $g(x)$ vanish nowhere in $[a,b]$. So that $\exists\ \sigma>0$ s.t $|g(x)|\ge\sigma$ $\forall x\in[a,b]$

claim: $\frac{f(x)}{g(x)}$ is absolutely cont on $[a,b]$

$$\sum\left|\frac{1}{g(b_k)}-\frac{1}{g(a_k)}\right| = \sum\frac{|g(b_k)-g(a_k)|}{|g(b_k)\cdot g(a_k)|} < \frac{\varepsilon}{\sigma^2}\ (=\varepsilon'), \text{ we get}$$

$$\sum\left|\frac{1}{g(b_k)}-\frac{1}{g(a_k)}\right|<\varepsilon' \Rightarrow \frac{1}{g(x)} \text{ is absolutely cont on } [a,b].$$

by conclusion, we get the result. (Any other suitable way is also considered)

i) Let $\varepsilon>0$ be given. Then there is a $\delta>0$ s.t. for every measureable set $A\subset[a,b]$ with $m(A)<\delta$, $\int_A|f|<\varepsilon$. Since the integrability of f implies that of $|f|$. Thus, for any finite collection $\{(x_i,x_i'): i=1,2,\dots N\}$ of pairwise open intervals in $[a,b]$ with $\sum(x_i'-x_i)<\delta$, we have

$$\sum\left|\int_{x_i}^{x_i'} f(t)\,dt\right| \le \sum\int_{x_i}^{x_i'}|f(t)|\,dt<\varepsilon$$

$$\Rightarrow \sum|F(x_i')-F(x_i)|<\varepsilon.$$

j) The class of Baire sets is defined to be the smallest σ-algebra $\mathcal{B}$ of subsets of X s.t each fn in $C_c(X)$ is measurable w.r.t $\mathcal{B}$. (and explanation need for $C_c(X)$).

Q2(a) Let (A, B) be a Hahn decomposition of X with respect to the signed measure ν so that ν^+ and ν^- are given by $\nu^+(E) = \nu(E \cap A)$ and $\nu^-(E) = -\nu(E \cap B)$ for any $E \in \mathcal{A}$

Now if $E \in \mathcal{A}$ is such that $\nu^-(E) = 0$ then for any $F \subset E$ with $F \in \mathcal{A}$ we have $0 \le \nu^-(F) \le \nu^-(E) = 0 \implies \nu^-(F) = 0$.

Therefore $\nu(F) = \nu^+(F) \ge 0$. Thus every measurable subset F of E is such that $\nu(F) \ge 0 \implies E$ is positive set

Again if $E \in \mathcal{A}$ such that $|\nu|(E) = 0$ then $\nu^+(E) + \nu^-(E) = 0$ and therefore $\nu^+(E) = \nu^-(E) = 0$. For any measurable set F with $F \subset E$, we have $0 \le \nu^+(F) \le \nu^+(E) = 0$ and $0 \le \nu^-(F) \le \nu^-(E) = 0$

showing $\nu^+(F) = \nu^-(F) = 0$ and hence $\nu(F) = \nu^+(F) - \nu^-(F) = 0$.

Thus $\nu(F) = 0$ for every measurable subset F of $E \implies$

E is null set.

b) Let $\langle A_n \rangle$ be a sequence of positive sets in X. Let $A = \cup A_i$.

Also let B be any measurable subset of A.

To prove that A is a positive set.

Write $B_n = B \cap A_n \cap A'_{n-1} \cap \dots \cap A'_1 \quad \forall n \in \mathbb{N}$

We know that if a set is measurable then its complement is also measurable. Also a countable intersection of measurable sets is measurable. These facts lead to the conclusion that B_n is a measurable subset of the positive set A_n and hence $\nu(B_n) \ge 0$, From the construction of B_n, it is clear that sets B_n are disjoint. Moreover

$$B = \cup B_n \implies \nu(B) = \cup \nu(B_n) \ge 0$$

Thus we have shown that

1) A is a measurable set. For A_n is a positive set
$\implies A_n$ is a measurable set.
$\implies \cup A_n$ is measurable $\implies A$ is measurable

2) $\forall B \subset A$ s.t B is measurable set $\implies \nu(B) \ge 0$.

By definition, this proves that A is a positive set.

3 Hahn Decomposition theorem: Suppose ν is a signed measure on a measurable space $(X, \mathcal{A})$. Then $\exists$ a positive set P and a negative set Q s.t. $P \cap Q = \emptyset$, $X = P \cup Q$.

Let A be a σ-algebra of subsets of X. Let ν be a signed measure on a measurable space $(X, \mathcal{A})$. Since ν assumes atmost one of the values $-\infty$ and ∞. without loss of generality we can suppose that ν does not take $-\infty$. Consider the family B of all negative subsets of X and let.

$$\lambda = \inf\{\nu(E) : E \in B\} \qquad \text{—} ①$$

Then $\exists$ a seq. $\langle E_n \rangle$ in B s.t $\lim_{n\to\infty} \nu(E_n) = \lambda$.

B is a family of negative set $\Leftrightarrow$ $\langle E_n \rangle$ is a seq. of -ve sets
$\Rightarrow \bigcup E_i$ is a negative set
$\Rightarrow Q$ is a -ve set on taking
$Q = \bigcup E_i$

Thus Q is a -ve set of X. Then ①, $\nu(Q) \geq \lambda$.

Next we consider the subset $Q - E_n$ of Q.

$$\because Q = (Q - E_n) \cup E_n$$

$$\therefore \nu(Q) = \nu(Q - E_n) + \nu(E_n) \leq \nu(E_n)$$

$$\nu(Q) \leq \nu(E_n) \quad \forall n \in N$$

$$E_n \in B \quad \forall n \in N.$$

$$\Rightarrow \nu(Q) \leq \lambda.$$

Thus we have shown that

$$\nu(Q) \leq \lambda \text{ and } \nu(Q) \geq \lambda.$$

This $\Rightarrow \nu(Q) = \lambda \Rightarrow \lambda > -\infty$.

Next our aim is to show that $P = X - Q$ is a +ve subset of X. Suppose the contrary. Then P is not positive and so P is negative. Hence by definition; for every measurable set $E \subset P$, $\nu(E) < 0$. Now E is a measurable subset of X with -ve measure. If E is a measurable set of finite negative measure.

consequently

$$\nu(A\cup Q) \geqslant \lambda \quad \text{by } ①$$

But $\quad \lambda \leq \nu(A\cup Q) = \nu(A) + \nu(Q) = \nu(A)+\lambda$

$\lambda \leq \nu(A)+\lambda \Rightarrow \nu(A) \geqslant 0$, a $\circledast$

Therefor $\nu(A) \geq 0$.

Thus $\quad P = X - Q$ is positive and Q is negative.

$\therefore X = P\cup Q \;;\; P\cap Q = \phi$.

4. a) By Riesz Representation theorem that if E is any measureble set of σ-finite measure then there is a unique $E \in L^{q}$ vanshing out side E, such that

$$F(f) = \int g_E f \, d\mu, \text{ for each } f \in L^{q} \text{ which}$$

vanishes out side E.

Moreaver, $\|g_E\|_q < \|F\|$.

Since g_E is unique and for $A \subset E$ we have $g_A = g_E$ a.e on A.

For each E of σ-finite measure set $\lambda(E) = \int |g_E|^{q} d\mu$.

Then for $A \subset E$, we have

$$\lambda(A) \leq \lambda(E) = \|g_E\|^{q}$$

Hence $\quad \lambda(A) \leq \lambda(E) \leq \|F\|_q$.

Let $\langle E_n \rangle$ be a ~~seq~~ of sets of σ-finite measure such that $\lambda(E_n)$ tends to the maximum value m of λ.

Then $H = \cup E_n$ is a set of σ-finite measure. and by the monotonocity of λ we have $\lambda(H) \geqslant \lambda(\cup E_n) = \Sigma \lambda E_n = m$.

Let g be defined by $g(x) = g_H(x) \quad \forall\, x \in H$

$= 0$

Then $g \in L^{q}$ as $g_H \in L^{q}$. If $H \subset E$, the $g_E = g_H$ a.e on H

And $\quad \int |g_E|_q = \lambda(E) \leq \lambda(H) = \int |g_H|^{q}$

and so $g_E = 0$ a.e on E. Thus $g_E = g$ almost every where on E.

Thus. $\quad F(f) = \int f g_E \, d\mu = \int f g \, d\mu$.

b) Lebesgue dominated convergence theorem :

Let g be an integrable function on E and let $\langle f_n \rangle$ be a ~~seq.~~ of measurable f^ns such that $|f_n| \leq g$ on E and

$\lim f_n = f$ a.e. on E. Then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

: clearly each f_n is integrable over E. Further, it follows from $\lim_{n\to\infty} f_n = f$ a.e on E and $|f_n| \leq g$ on E that

$|f| \leq g$ a.e on E

Hence f is integrable over E. Consider a ~~seq.~~ $\langle h_n \rangle$ of ~~f^n~~ defined by $h_n = f_n + g$.

clearly, h_n is a nonnegative and integrable f^n for each n.

Therefore, by Fatou's Lemma, we get

$$\int f+g \leq \underline{\lim} \int_E (f_n + g)$$

$$\Rightarrow \quad \int_E f \leq \underline{\lim} \int_E f_n.$$

Similarly, consider a ~~seq.~~ $\langle k_n \rangle$ defined by $k_n = g - f_n$ and observe that k_n is a nonnegative and integrable f^n for each n. So, again by Fatou's Lemma, we have

$$\int_E (g-f) \leq \underline{\lim} \int_E (g - f_n) \Rightarrow \int_E f \geq \overline{\lim} \int_E f_n.$$

Q.5 It is sufficient to prove that the set $I(D^{-}f > \alpha)$ is measurable for any real number α. Consider, for a real number α, the set

$$A = \{x \in I,\ D^{-}f(x) > \alpha\}$$

corresponding to each pair of positive integer m and n, we denote by A_{mn} the union of all intervals $[x_1, x_2]$ which are sub-sets of I and have the properties

$$|x_2 - x_1| < \frac{1}{m} \text{ and } \frac{f(x_2) - f(x_1)}{x_2 - x_1} > \alpha + \frac{1}{n}.$$

clearly, the set A_{mn} is measurable and so is the set $\bigcup_n \bigcap_m A_{mn}$.

Let $c \in A$. Then $\exists$ an integer n_0 such that $D^- f(c) > \alpha + \frac{1}{n_0}$.

Therefore, corresponding to each value of m, there exists a point x^* of I satisfying

$$0 < |x^* - c| < \frac{1}{m} \text{ and } \frac{f(x^*) - f(c)}{x^* - c} > \alpha + \frac{1}{n_0}.$$

Then $c \in \bigcup_n \bigcap_m A_{mn}$ as the closed interval $[c, x^*]$ is, contained in A_{mn} where $c \in A_{mn_0}$ for each m. This further gives that $A \subset \bigcup \bigcap A_{mn}$.

To get the reverse inequality, let $c \in \bigcup \bigcap A_{mn}$. There is an n_0 such that $c \in \bigcap A_{mn_0}$, where $c \in A_{mn_0}$ $\forall m$. This gives, corresponding to each m, an interval $[x_1, x_2]$ such that $x_1 \le c \le x_2$.

$$|x_2 - x_1| < \frac{1}{m} \text{ and } \frac{f(x_2) - f(x_1)}{x_2 - x_1} > \alpha + \frac{1}{n_0}, \quad \text{— (1)}$$

Suppose $x_1 < c < x_2$. Then in view of (1), at least one of the following two must be true.

(a) $\frac{f(x_1) - f(c)}{x_1 - c} > \alpha + \frac{1}{n_0}$.

(b) $\frac{f(x_2) - f(c)}{x_2 - c} > \alpha + \frac{1}{n_0}$.

Further, by (1) we see that if $x_2 = c$, then (a) is true, and if $x_1 = c$ then (b) is true. In either case, $\exists$ a no. $x \in I$ for which

$$0 < |x - c| < \frac{1}{m} \text{ and } \frac{f(x) - f(c)}{x - c} > \alpha + \frac{1}{n_0}.$$

This implies that $D^- f(c) \geq \alpha + \frac{1}{n_0} \geq \alpha$. Thus $c \in A$.

Hence $\bigcup \bigcap A_{mn} \subset A$.

(b) For finding $D^+ f(0) = 1$, $D_+ f(0) = -1$, $D^- f(0) = 5$ and $D_- f(0) = 3$

(calculation part is needed)

Q.6 (a) Jordan Decomposition Theorem

A function f defined on $[a,b]$ is of bounded variation if and only if it can be expressed as a difference of two monotone increasing $\cancel{fn}$ on $[a,b]$.

Ans: Let f be of bounded variation on $[a,b]$. Define g and h by $g = \frac{1}{2}[Vf + f]$ and $h = \frac{1}{2}[Vf - f]$. so that $f = g - h$.

Now, if x_1, x_2 is any pair of points in $[a,b]$ with $x_2 > x_1$, then

$$g(x_2) - g(x_1) = \frac{1}{2}\left[\{Vf(x_2) - Vf(x_1)\} + \{f(x_2) - f(x_1)\}\right]$$

$$h(x_2) - h(x_1) = \frac{1}{2}\left[\{V_f(x_2) - V_f(x_1)\} - \{f(x_2) - f(x_1)\}\right].$$

But, f being of bounded variation on $[a,b]$, in particular on $[x_1, x_2]$ we have

$$|f(x_2) - f(x_1)| \le T_{x_1}^{x_2}(f) = V_f(x_2) - V_f(x_1).$$

Hence $g(x_2) \ge g(x_1)$ and $h(x_2) \ge h(x_1)$ verifying that g and h both are monotone increasing $\cancel{fn}$.

Conversely, if $f = g - h$ where g and h are monotone increasing $\cancel{fn}$s then for any partition P of $[a,b]$, we have

$$\Sigma\, |f(x_i) - f(x_{i-1})| \le \Sigma\,[g(x_i) - g(x_{i-1})] + \Sigma\,[h(x_i) - h(x_{i-1})]$$

$$= g(b) - g(a) + h(b) - h(a)$$

$$\Rightarrow T_a^b(f) < \infty.$$

Hence f is a $\cancel{fn}$ of bounded variation.

(b) Since f is bounded and measurable, it is integrable. F is a continuous $\cancel{fn}$ of bounded variation on $[a,b]$ and hence F' exists a.e in $[a,b]$.

Let $|f| \le K$. Consider the set

$$f_n(x) = \frac{F(x+h) - F(x)}{h} \quad \text{with } h = \frac{1}{n}. \text{ Then}$$

$$f_n(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt \Rightarrow |f_n| \le K.$$

Also, $f_n \to F'$ a.e. If $c \in [a,b]$ is arbitrary, then the bounded convergence theorem implies that

$$\int_a^c F'(x)\,dx = \lim \int_a^c f_n(x)\,dx$$

$$= \lim \frac{1}{h}\int_a^c [F(x+h)-F(x)]\,dx$$

$$= \lim \left[\frac{1}{h}\int_c^{c+h} F(x)\,dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx\right].$$

But F being a continuous ~~fn~~, we ~~do~~ note that

$$\lim \frac{1}{h}\int_c^{c+h} F(x)\,dx = \lim \frac{1}{h}\, k \int_c^{c+h} F(x)\,dx$$

$$= \lim F(c+\theta h) = F(c) \qquad (0<\theta<1)$$

and similarly $\lim \frac{1}{h}\int_a^{a+h} F(x)\,dx = F(a)$.

Thus $\int_a^c F'(x)\,dx = F(c) - F(a) = \int_a^c f(x)\,dx$

This implies that $\int_a^c [F'(x) - f(x)] = 0$

This is true for all $c \in [a,b]$.

Q.7. ~~(i)~~ If f is absolutely continuous on $[a,b]$ and $f'=0$ a.e.,

(i) show that f is constant ~~fn~~.

Let $c \in [a,b]$ be arbitrary. Then it is enough to prove that $f(c) = f(a)$. Let $E = \{x \in (a,c) : f'(x) = 0\}$. Then $E \subset (a,c)$.

Let ε and η be arbitrary positive nos. Then, each $x \in E$ is the left end point of an arbitrary small interval $[x, x+h] \subset [a,c]$ such that $|f(x+h) - f(x)| < \eta h$ $(h>0)$.

Thus, the set of all such intervals forms a Vitali cover of E.

From among these intervals, we can choose a finite collection $\{[x_i, y_i] ; i = 1, 2, \ldots, N\}$ of pairwise disjoint intervals which covers all of E except for a set of measure less than δ, where $\delta > 0$ is the number corresponding to $\varepsilon > 0$ in the definition of the absolute continuity of f.

we have

$$y_0 = a \le x_1 < y_1 \le x_2 < y_2 \dots < y_N \le c = x_{N+1}$$

Also, the complement of the union of these intervals is again the union of a finite number of disjoint intervals $\{[y_i, x_{i+1}] : i = 0, 1, \dots N\}$. Their combined length is not greater than δ,

$$\Sigma \, |x_{i+1} - y_i| < \delta .$$

Now, we note that

$$\Sigma \, | f(y_i) - f(x_i) | \le \eta \sum_{i=1}^{N} (y_i - x_i)$$
$$< \eta (c-a).$$

and $$\Sigma \, | f(x_{i+1}) - f(y_i) | < \varepsilon .$$

by the absolute continuity of f. Hence

$$| f(c) - f(a) | = \left| \Sigma [f(x_{i+1}) - f(y_i)] + \Sigma (f(y_i) - f(x_i)) \right|$$
$$\le \varepsilon + \eta (c-a).$$

since ε and η are arbitrary positive nos, the result follow by letting $\varepsilon \to 0$ and $\eta \to 0$.

(ii) For $\varepsilon = 1$, there is a $\delta > 0$ such that for every finite collection $\{ [x_i, x_i'), i = 1, 2, \dots n \}$ of pairwise disjoint intervals in $[a, b]$ with $\Sigma |x_i' - x_i| < \delta$, we have

$$\Sigma \, | f(x_i') - f(x_i) | < 1.$$

select a natural number $N > \dfrac{b-a}{\delta}$.

Divide $[a, b]$ by means of points

$$a = c_0 < c_1 < c_2 < \dots < c_N = b$$

such that $c_j - c_{j-1} < \delta$, for $j = 1, 2, \dots N$. Therefore, for every finite collection $\{[x_i, x_i')\}$ of pairwise disjoint subintervals in $[c_{j-1}, c_j]$ we have

$$\Sigma \, | f(x_i') - f(x_i) | < 1 \Rightarrow T_{c_{j-1}}^{c_j} (f) \le 1 \, , \; j = 1, 2, \dots N$$

Hence $$T_a^b (f) = \sum_{j=1}^{N} T_{c_{j-1}}^{c_j} (f) \le N < \infty .$$

Q.8. A measure μ defined on the σ-algebra of Baire sets is called a Baire measure if it is finite for each compact Baire set.

Since E is σ-bounded, it is contained in a σ-compact open set O.

If $\mu E = \infty \implies \mu O = \infty$.

In this case nothing to prove.

If E is compact G_δ, then there is a function $\varphi \in C_c(X)$ which is 1 on E and $0 \le \varphi < 1$ on $\tilde{E}$.

Let $O_n = \{x : \varphi(x) > 1 - \frac{1}{n}\}$. Then each O_n is a σ-compact open set, $O_n \supset O_{n+1}$, and $E = \cap O_n$.

Since $\bar{O}_1$ is compact, $\mu O_1 < \infty$ and $\mu E = \lim \mu O_n$.

Thus for some O_n we have $\mu O_n < \mu E + \varepsilon$.

Let $E = E_1 \cap \tilde{E}_2$ where E_1 and E_2 are compact G_δ's with $E_2 \subset E_1$, and let U be a σ-compact open set with $\bar{U}$ compact such that $E_1 \subset U$ and $\mu U < \mu E_1 + \varepsilon$.

Set $O = U \cap \tilde{E}_2$. Then O is the intersection of F_σ's and so O is an F_σ. Since O is contained in the compact set $\bar{U}$, O must be σ-compact. Since $O \sim E = U \sim E_1$, we have $\mu(O \sim E) = \mu(U \sim E_1) = \mu U - \mu E_1 < \varepsilon$. Thus $\mu O < \mu E + \varepsilon$, then proof is completed for sets E in the semi-algebra C generated by the compact G_δ's.

(And we consider the proof if E is σ-algebra case also).